

Charpit's method is a general method of solving the p. d. e. of order one.

Charpit's Method:— Let the p. d. e. be

$$f(x, y, z, p, q) = 0 \dots\dots\dots (1)$$

Also we have

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy$$

$$\text{i.e } dz = p dx + q dy \dots\dots\dots (2)$$

Consider another relation

$$F(x, y, z, p, q) = 0 \dots\dots\dots (3)$$

such that the values of p and q obtained by solving (1) and (3) and are substituted in (2) then it becomes integrable. Thus z, p, q may be expressed as function of x and y. The integral of (2) is the complete integral of (1).

Now differentiating (1) and (3) w. r. t. x, we get

$$\left. \begin{aligned} \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} &= 0 \text{ and} \\ \frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} &= 0 \end{aligned} \right\} \left[\because \frac{\partial z}{\partial x} = p \right]$$

Again differentiating w. r. t. y, we get

$$\left. \begin{aligned} \frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \\ \frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} + \frac{\partial F}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial y} = 0 \end{aligned} \right\} \text{ and}$$

$$\left. \begin{aligned} \frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} \\ \frac{\partial F}{\partial x} + p \frac{\partial F}{\partial z} + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} \end{aligned} \right\}$$

Eliminating $\frac{\partial p}{\partial x}$ from the first pair, we obtain

$$\left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} + \frac{\partial f}{\partial q} \cdot \frac{\partial q}{\partial x} \right) \frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot p + \frac{\partial F}{\partial q} \cdot \frac{\partial q}{\partial x} \right) \cdot \frac{\partial f}{\partial p} = 0$$

$$\Rightarrow \left(\frac{\partial f}{\partial x} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial x} \cdot \frac{\partial f}{\partial p} \right) + \left(\frac{\partial f}{\partial z} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial p} \right) p + \left(\frac{\partial f}{\partial q} \cdot \frac{\partial F}{\partial p} - \frac{\partial F}{\partial q} \cdot \frac{\partial f}{\partial p} \right) \frac{\partial q}{\partial x} = 0 \dots \dots \dots (4)$$

Similarly, eliminating $\frac{\partial q}{\partial y}$ between the last pair we have

$$\left(\frac{\partial f}{\partial y} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial y} \cdot \frac{\partial f}{\partial q} \right) + \left(\frac{\partial f}{\partial z} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial z} \cdot \frac{\partial f}{\partial q} \right) q$$

$$+ \left(\frac{\partial f}{\partial p} \cdot \frac{\partial F}{\partial q} - \frac{\partial F}{\partial p} \cdot \frac{\partial f}{\partial q} \right) \cdot \frac{\partial p}{\partial y} = 0 \dots \dots \dots (5)$$

$$\therefore \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} = \frac{\partial^2 z}{\partial x \cdot \partial y}$$

Here, the last term of (4) is the same as the last term in (5) except for the minus sign.

Hence we add (4) and (5) the last terms will be vanish. Therefore, adding (4) and (5) and

rearranging the terms, we get

$$\left(\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z} \right) \frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z} \right) \frac{\partial F}{\partial q} + \left(-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q} \right) \frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p} \right) \cdot \frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q} \right) \frac{\partial F}{\partial y} = 0 \dots \dots \dots (6)$$

which is the linear eqn. of the first order with x, y, z, p, q as independent variables and F dependent variable.

The Charpit's auxiliary eqns. are

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}}$$

$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{0} \dots \dots \dots (7)$$

Any integral of (7) will satisfy (6). The simplest relation involving at one of p and q may be taken as $F = 0$. Now from $F = 0$ and $f = 0$ the values of p and q should be found in terms of x and y and should be substituted in (2) which on integration gives the solution.

Q. 1) Solve: $px + qy = pq$ by applying Charpit's method.

Solution:- Here p. d. e. is

$$f(x, y, z, p, q) = px + qy - pq = 0$$

$$\text{So that } \frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = q, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = x - q,$$

$$\frac{\partial f}{\partial q} = y - p.$$

The Charpit's auxiliary equations are

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$$= \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dx}{-(x-q)}$$

$$= \frac{dy}{-(y-p)} = \frac{dF}{0}$$

Taking the first two fractions, we have

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log a$$

$\Rightarrow p = aq$ [where a is arbitrary constant]

Putting $p = aq$ in the given equation, we get

$$aqx + qy - aq^2 = 0 \Rightarrow ax + y - aq = 0$$

$$\Rightarrow ax + y = aq = p \text{ i.e. } p = ax + y$$

Putting the values $p = ax + y$ and $q = \frac{1}{a}(ax + y)$

in $dz = p dx + q dy$, we get

$$dz = (ax + y) dx + \frac{1}{a}(ax + y) dy = (ax + y) \left[dx + \frac{1}{a} dy \right]$$

$$\Rightarrow a dz = (a dx + dy)(ax + y)$$

Integrating, $az = \frac{(y + ax)^2}{2} + b$ (constant)

Which is the complete solution. where a and b are arbitrary constants.

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